Canonical Deformed Groups of Diffeomorphisms and Finite Parallel Transports in Riemannian Spaces

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Abstract

We show that finite parallel transports of vectors in Riemannian spaces, determined by the multiplication law in the deformed groups of diffeomorphisms, and sequences of infinitesimal parallel transports of vectors along geodesics are equivalent.

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Performing group-theoretic description of spaces with torsion-free affine connection and Riemannian spaces with the help of deformed groups of diffeomorphisms Γ_T^H, in work [1] we introduced the concept of parallel transport of vectors to finite and not just infinitesimal distance $\tilde{t} = x' - x$, the concept following from multiplication law in groups Γ_T^H . On the other hand, in classic approach finite parallel transports are generated from infinitesimal ones in case of movement along curves [2], in curved space the result depending upon the selected curve. In case of infinitesimal shifts $\tilde{t}$ both transports yield identical results, affine vielbein field and affine connection coefficients, defined by the action of deformed group of diffeomorphisms Γ_T^H , are defined by the first two orders in expansion of functions of deformations with respect to $\tilde{t}$, while all higher orders in spaces with affine connection remain undefined though influence the result of finite parallel transports.

In this work we show that requirement of that for two arbitrary points x and x' in arbitrarily curved space there exists at least one curve connecting them, for which sequence of infinitesimal parallel transports would yield result identical to finite parallel transport (specified by multiplication law of deformed group of diffeomorphisms Γ_T^H) is enough to prove that such curve is (locally) unique and is a geodesic. In this approach the derivatives from the deformation functions in the direction of geodesic are the first integrals of system of differential equations which specify it. By the latter deformation functions (and not just their first two orders in expansion with respect to shifts $\tilde{t}$) are uniquely defined with respect to affine vielbein field and affine connection coefficients, thus finite parallel transport to the distance $\tilde{t} = x' - x$ specified by multiplication law of group Γ_T^H is unique and yields the result of parallel transport along the geodesic connecting points x and x' .

Groups Γ_T^H produced by means of such deformations are a certain generalization of finite parametric canonical Lie groups [3] for infinite case and, therefore, are also called canonical as well as to deformations with the help of which they are built.

In the Riemannian space deformation functions (including their second order in expansion with respect to the shifts specified by connection) are completely defined with respect to the orthonormal vielbein field by requirement for vectors not to change their length and only rotate in case of finite parallel transports in group-theoretic mode. This work shows that such deformations are canonical.

We also show that in Riemannian space, where geodesics in natural parameterization are extremals of energy functional which is defined by metric, the function of central extremal fields, which start from each point [4] (action function) uniquely (with accuracy to the choice of orthonormal vielbein field coordinated with metric) define deformation functions. Moreover, Hamilton-Jacobi equation, which is satisfied by action function, for deformation function comes down to the equation which follows from the requirement of invariability of vector length in case of finite parallel transports.

The determined relation between the first integrals of geodesics as well as action function with deformation functions results in their new understanding and can be of applied significance, particularly in gravitation theory.

This article uses the same notations and assumptions as [1]. Specifically, all relations are obtained with the bounds of single coordinate chart with fixed though arbitrary coordinates.

1 Deformed groups of diffeomorphisms

Let in coordinate chart O with coordinates x^μ (coordinate indices are selected from Greek alphabet) there act a local group of diffeomorphisms in additive parameterization $\Gamma_T = \{\tilde{t}(x)\}$ (*undeformed group of diffeomorphisms*) [1] according to the formula

$$x'^\mu = x^\mu + \tilde{t}^\mu(x).$$

Smooth functions $\tilde{t}^\mu(x)$ parameterizing group Γ_T satisfy the requirement $\{\delta^\mu_\nu + \partial_\nu \tilde{t}^\mu(x)\} \neq 0$, $\forall x \in O$, where $\partial_\nu := \partial/\partial x^\nu$, and multiplication law in it $\tilde{t}'' = \tilde{t} \times \tilde{t}'$ is specified by the formula:

$$\tilde{t}''^\mu(x) = \tilde{t}^\mu(x) + \tilde{t}'^\mu(x'). \tag{1}$$

Deformed group of diffeomorphisms $\Gamma^H_T = \{t(x)\}$ [1] is parameterized by the functions $t^m(x)$ (we will use indices from Latin alphabet for them) is produced from group Γ_T as isomorphism specified by deformation H according to the formula

$$t^m(x) = H^m(x, \tilde{t}(x)) \tag{2}$$

with the help of smooth deformation functions $H^m(x, \tilde{t}(x))$ with properties

1H) $H^m(x, 0) = 0,\ \forall x \in O$;

2H) $\exists$ functions $K^\mu(x, t(x)) : K^\mu(x, H(x, \tilde{t}(x))) = \tilde{t}^\mu(x),\ \forall x \in O,\ \tilde{t} \in \Gamma_T$.

Functions $t^m(x)$ satisfy the condition det $\{\delta^\mu_\nu + d_\nu K^\mu(x, t(x))\} \neq 0,\ \forall x \in O$, where $d_\nu := d/dx^\nu$. Functions $K^\mu(x, t(x))$ which are present in property 2H) specify reverse transition $\Gamma^H_T \to \Gamma_T$. Multiplication law $t'' = t * t'$ in group Γ^H_T is defined by multiplication law (1) in group Γ_T and isomorphism (2):

$$t''^m(x) = \varphi^m(x, t(x), t'(x')) := H^m(x, K(x, t(x) + K(x', t'(x'))), \tag{3}$$

where

$$x'^\mu = f^\mu(x, t(x)) := x^\mu + K^\mu(x, t(x)). \tag{4}$$

Group Γ^H_T acts smoothly in chart O according to the formula (4).

With the help of functions $\varphi(x,t,t')$ which specify multiplication law (3) auxiliary matrices are defined:

$$\lambda(x,t)^m_n := \partial_{n'}\varphi^m(x,t,t')|_{t'=0} = h(x+\tilde{t})^\mu_n \partial_{\tilde{\mu}} H^m(x,\tilde{t})|_{\tilde{t}=K(x,t)}, \tag{5}$$

$$\mu(x,t)^m_n := \partial_{n'}\varphi^m(x,t',t)|_{t'=0} = h(x)^\nu_n(\delta^\mu_\nu + \partial_\nu \tilde{t}^\mu)\partial_{\tilde{\mu}} H^m(x,\tilde{t})|_{\tilde{t}=K(x,t)}, \tag{6}$$

where $h(x)^\mu_m := \partial_m K^\mu(x,t)|_{t=0}$ and $\partial_m := \partial/\partial t^m$ (primed index standing for differentiation with respect to t' and tilded one for that with respect to $\tilde{t}$).

Let's emphasize that despite the fact that deformations are defined as isomorphisms of diffeomorphisms group, in general case they cannot be compensated by coordinate transformations in chart O which result only in internal automorphisms of diffeomorphisms group.

As shown in work [1], deformed group of diffeomorphisms Γ^H_T specifies by its action in chart O geometric structure of space with torsion-free affine connection and deformations lead to change of its characteristics, specifically to curvature.

Generators $X_m = h(x)^\mu_m \partial_\mu$ of action (4) of group Γ^H_T specify in O an affine vielbein field, matrices $h(x)^\mu_m$ and matrices $h(x)^m_\mu$ reverse to them transit between coordinate and affine veilbeins.

Element of group Γ^H_T are accorded with vector fields $t = t^m(x)X_m$ and parameters $t^m(x)$ of group Γ^H_T appear for components of these fields in basis X_m .

Multiplication law (3) in group Γ^H_T written down for elements t and τ for the case of infinitesimal second multiplier

$$(t * \tau)^m(x) = t^m(x) + \lambda(x,t(x))^m_n \tau^n(x'),$$

defines the rule of composition of vectors fitted in different points or the *rule of parallel transport* of vector field τ from point x' to point x for finite distance $x' - x = \tilde{t} = K(x,t)$:

$$\tau^m_{\|}(x) = \lambda(x,t)^m_n \tau^n(x').$$

In coordinate basis this relation appears as

$$\tau^\mu_{\|}(x) = \lambda(x,\tilde{t})^\mu_\nu \tau^\nu(x') = \partial_{\tilde{\nu}} H^\mu(x,\tilde{t})\tau^\nu(x'), \tag{7}$$

(here, in particular, $H^\mu(x,\tilde{t}) = h(x)^\mu_m H^m(x,\tilde{t})$) or in case of infinitesimal $\tilde{t}$:

$$\tau^\mu_{\|}(x) = \tau^\mu(x) + \tilde{t}^\nu(x)\nabla_\nu \tau^\mu(x),$$

where

$$\nabla_\nu \tau^\mu(x) = \partial_\nu \tau^\mu(x) + \Gamma(x)^\mu_{\sigma\nu}\tau^\sigma(x)$$

is a covariant derivative while functions

$$\Gamma(x)^\mu_{\sigma\nu} := \partial^2_{\tilde{\sigma}\tilde{\nu}} H^\mu(x,\tilde{t})|_{\tilde{t}=0} \tag{8}$$

acquire geometric meaning of (torsion-free) affine connection coefficients in coordinate basis. The higher orders of expansion of function $H^\mu(x,\tilde{t})$ with respect to $\tilde{t}$ do not influence the connection though, as follows from formula (7), define the result of finite parallel transports thus require precise definition which is given in the next section of the work. The skew-symmetric part

$$R^m_{lkn} := \rho^m_{lkn} - \rho^m_{lnk} \tag{9}$$

of coefficients $\rho^m_{lkn} := \partial^2_{lk}\mu(x,t)^m_n|_{t=0}$ (here and later dependence upon x is not explicitly shown for cases where it is obvious) written down in coordinate basis appears as [1]:

$$R^\mu_{\lambda k\nu} = \partial_k \Gamma^\mu_{\nu\lambda} - \partial_\nu \Gamma^\mu_{k\lambda} + \Gamma^\mu_{k\sigma}\Gamma^\sigma_{\nu\lambda} - \Gamma^\mu_{\nu\sigma}\Gamma^\sigma_{k\lambda}, \tag{10}$$

consequently it is Riemann-Cristoffel curvature tensor. In Riemannian space deformation functions are satisfied by equations

$$\partial_{\tilde{\mu}} H^\rho(x,\tilde{t})\partial_{\tilde{\nu}} H^\sigma(x,\tilde{t}) g(x)_{\rho\sigma} = g(x+\tilde{t})_{\mu\nu}, \tag{11}$$

which follows from the requirement for vectors not to change their length and just rotate in case of finite parallel transports. Equation (11) allows to uniquely define deformation functions in coordinate basis $H^\mu(x,\tilde{t})$ according to metric, specifically their second order in expansion with respect to $\tilde{t}$, i.e. according to (8) affine connection coefficients which become equal to Cristoffel symbols:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}). \tag{12}$$

2 Canonical deformations

Let's assume that group Γ^H_T is such that between two arbitrary points x , $x' \in O$ there exists parametric curve $x(s),\ s \in I = [0,T] :\ x(0) = x,\ x(T) = x'$, along which all tangent vectors $\tau(s) := \dot{x}(s)$ (point stands for differentiation with respect to s) parallel to each other in terms of parallel finite transports, this is according to (7)

$$\tau^\mu(s) = \lambda(x(s), \tilde{t}(s,s'))^\mu_\nu \tau^\nu(s'),\ \forall s,\ s' \in I, \tag{13}$$

where $\tilde{t}(s,s') := x(s') - x(s)$. Condition (13) for deformation function becomes:

$$\tau^\mu(s) = \partial_{\tilde{\nu}} H^\mu(x(s), \tilde{t}(s,s'))\tau^\nu(s') = \frac{d}{ds'} H^\mu(x(s), \tilde{t}(s,s')),\ \forall s,\ s' \in I. \tag{14}$$

Definition. *Deformations are called* ***canonical*** *and groups Γ^H_T produces with their help are called* ***canonical deformed groups of diffeomorphisms****, provided that between two arbitrary points $x,\ x' \in O$, there exists at least one curve $x(s)$, along which deformation functions satisfy the requirement (14).*

Differentiation of equation (14) with respect to s' with $s' = s$, with consideration of formula (8) yields the equation of curve $x(s)$, the existence of which is assumed in Definition:

$$\dot{\tau}^\mu(s) + \Gamma(x(s))^\mu_{\sigma\nu}\tau^\sigma(s)\tau^\nu(s) = 0,\ \dot{x}^\mu(s) = \tau^\mu(s), \tag{15}$$

which is evidence of the fact that curve $x(s)$ is geodesic in affine parameterzation for the structure of affine connection specified in O by the action of group Γ^H_T. Locally between two points x and x' only one geodesic can be drawn, accordingly *the requirement of existence of at least one curve for which condition (14) is satisfied is enough for identification that such curve is geodesic.*

Now let's rewrite conditions (13) and (14) for $s = 0$ (and substituting s' for s) with consideration of definition of shift $\tilde{t}(0, s) = x(s) - x$:

$$\tau^\mu = \lambda(x, x' - x)^\mu_\nu {\tau'}^\nu = H^\mu(x, x' - x) =: u^\mu(x, x', \tau'). \tag{16}$$

Here we assumed $\tau := \tau(0)$ as well as denoted current points and tangent vectors to curve as $x' := x(s),\ \tau' := \tau(s)$. Thus functions $u^\mu(x, x', \tau')$, being derivatives from deformation functions along geodesics, are constant for canonical deformation and are independent first integrals of autonomous system of differential equations of geodesics (15). Thus, functions $u^\mu(x, x', \tau')$ are to satisfy the equation [5]

$$\frac{\partial u^\mu}{\partial {x'}^\nu}{\tau'}^\nu - \frac{\partial u^\mu}{\partial {\tau'}^\nu}\Gamma(x')^\nu_{\rho\sigma}{\tau'}^\rho{\tau'}^\sigma = 0, \tag{17}$$

geodesics being characteristics of which. With consideration of relation (16) equation (17) comes down to the following equation for auxiliary matrices $\lambda(x, \tilde{t})^\mu_\nu$:

$$[\partial_{\tilde{\rho}}\lambda(x, \tilde{t})^\mu_\sigma - \Gamma(x + \tilde{t})^\nu_{\rho\sigma}\lambda(x, \tilde{t})^\mu_\nu]{\tau'}^\rho{\tau'}^\sigma = 0, \tag{18}$$

which in terms of deformation functions may become:

$$[\partial^2_{\tilde{\rho}\tilde{\sigma}}H^\mu(x, \tilde{t}) - \Gamma(x + \tilde{t})^\nu_{\rho\sigma}\partial_{\tilde{\nu}}H^\mu(x, \tilde{t})]{\tau'}^\rho{\tau'}^\sigma = 0. \tag{19}$$

Boundary conditions for auxiliary matrices $\lambda(x, \tilde{t})^\mu_\nu$ and deformation functions $H^\mu(x, \tilde{t})$ are specified in point x and follow from properties 1H), 2H):

$$\lambda(x, 0)^\mu_\nu = \delta^\mu_\nu, \tag{20}$$

$$H^\mu(x, 0) = 0,\ \partial_\nu H^\mu(x, 0) = \delta^\mu_\nu, \tag{21}$$

which in selection of initial vector τ defines boundary conditions for the first integrals:

$$u^\mu(x, x, \tau) = \tau^\mu. \tag{22}$$

According to the theorem of existence and uniqueness of solution to linear differential equations in partial derivatives [5], Cauchy problem (17), (22) for arbitrary symmetric with respect to the

lower indices and smooth functions $\Gamma(x)^{\nu}_{\rho\sigma}$ has unique solution which corresponds to geodesic starts from the point x in the direction of vector τ . This provides for unique solvability of problem (18), (20) for auxiliary matrices $\lambda(x,\tilde{t})^{\mu}_{\nu}$, as well as problem (19), (20) for deformation functions in coordinate basis $H^{\mu}(x,\tilde{t})$ which, accordingly, correspond to the central field of geodesics which start from point x with different initial vectors τ. *Thus, arbitrary (torsion-free) affine connection coefficients $\Gamma(x)^{\nu}_{\rho\sigma}$ for canonical deformations uniquely define deformation functions in coordinate basis.*

The zeroth and first orders of expansion of function $H^{\mu}(x,\tilde{t})$ with respect to shifts $\tilde{t}$ are defined by boundary conditions (21). Sequentially differentiating equation (19) with respect to s in zero we derive the following recurrent formula for finding coefficients of expansion of functions $H^{\mu}(x,\tilde{t})$ with respect to $\tilde{t}$ in any order $n \geq 2$:

$$\partial^{(n)}_{\tilde{\nu}_1,\ldots,\tilde{\nu}_n} H^{\mu}(x,0) = \partial^{(n-2)}_{\{\tilde{\nu}_1,\ldots,\tilde{\nu}_{n-2}}[\Gamma(x+\tilde{t})^{\sigma}_{\tilde{\nu}_{n-1}\tilde{\nu}_n\}}\partial_{\tilde{\sigma}} H^{\mu}(x,\tilde{t})]|_{\tilde{t}=0},$$

where curly brackets, as usual, stand for symmetrization with respect to indices placed in them. Specifically, with $n=2$ we derive relation (8) and with $n=3$:

$$\triangle^{\mu}_{\nu\rho\tau} = \partial_{\{\nu}\Gamma^{\mu}_{\rho\tau\}} + \Gamma^{\sigma}_{\{\nu\rho}\Gamma^{\mu}_{\tau\}\sigma},$$

where $\triangle^{\mu}_{\nu\rho\tau} := \partial^3_{\tilde{\nu}\tilde{\rho}\tilde{\tau}} H^{\mu}(x,\tilde{t})|_{\tilde{t}=0}$, wherefrom, as shown in [1], it follows that coefficients $\rho^{\mu}_{\nu\rho\tau}$ are defined by curvature tensor (10) uniquely:

$$\rho^{\mu}_{\nu\rho\tau} = \frac{1}{3}(R^{\mu}_{\nu\rho\tau} + R^{\mu}_{\rho\nu\tau}). \tag{23}$$

and definition (9) comes down to the known identity $R^{\mu}_{\nu\rho\tau} + R^{\mu}_{\rho\tau\nu} + R^{\mu}_{\tau\nu\rho} = 0$.

Let's summarize the derivations.

Proposition 1 (of uniqueness of canonical deformations).

a) The condition of canonicity of deformations (14) locally uniquely defines curve between points x and x' , along which it is performed, this curve being geodesic in affine parameterization of space with (torsion-free) affine connection, the structure of which is specified in O by the action of group Γ^H_T .

b) The functions of canonical deformations in coordinate basis $H^{\mu}(x,\tilde{t})$ are uniquely defined by coefficients $\Gamma(x)^{\sigma}_{\rho\sigma}$ of (torsion-free) affine connection in a single-valued manner, and derivatives from them along geodesics are the first integrals of system of differential equations for geodesics.

It is easy to verify that for three arbitrary points x, x' , and x'' which lie on one geodesic, matrix $\lambda(x,x'-x)^{\mu}_{\sigma}\lambda(x',x''-x')^{\sigma}_{\nu}$ do not depend upon x' and also satisfy the equation (18) and the condition (20). From uniqueness of solution to problem (18), (20), this is provided for by

composition law of parallel transport of arbitrary vectors θ along geodesic:

$$\theta^{\mu}(x) = \lambda(x, x'' - x)^{\mu}_{\nu}\theta^{\nu}(x'') = \lambda(x, x' - x)^{\mu}_{\sigma}\lambda(x', x'' - x')^{\sigma}_{\nu}\theta^{\nu}(x''), \tag{24}$$

which, in its turn, leads to coincidence of *arbitrary sequence of parallel vector transports along geodesic (including integral sequence of infinitesimal transports) with resulting finite transport.* And vice versa, direct verification shows that if composition law (24) is performed for three arbitrary points $x,\ x'$, and x'' on geodesic and vector θ for finite parallel transports, the functions $\lambda(x, x' - x)^{\mu}_{\nu}$, with the help of which the law is performed, satisfy the equation (18), accordingly, the *respective deformation will be canonic.*

Proposition 2 (of finite parallel transports).

For finite parallel transport of arbitrary vectors θ to the distance $\tilde{t} = x' - x$ to yield the result of integral sequence of infinitesimal parallel transports along geodesic connecting points x and x' , it is necessary and sufficient for deformed group of diffeomorphisms Γ^{H}_{T} , defining the transport, to be canonic.

Let's point out that the condition of canonicity of deformations (14) is applied to the functions of deformation in coordinate vielbein $H^{\mu}(x, \tilde{t})$, accordingly the selection of affine veilbein X_m remains arbitrary and does not influence upon the canonicity of deformation.

That is, if deformation with deformation functions $H^m(x, \tilde{t})$ is canonic, canonic is the deformation with functions

$$H^{'m}(x, \tilde{t}) = L(x)^m_n H^n(x, \tilde{t})$$

where $L(x)^m_n$ are arbitrary nondegenerated matrices dependent upon x.

3 Criteria of canonicity

Let's study the parameters of deformed group of diffeomorphisms Γ^{H}_{T} which correspond to shifts $\tilde{t}(s, s') := x(s') - x(s)$ along curve $x(s)$:

$$t^m(s, s') = H^m(x(s), \tilde{t}(s, s')). \tag{25}$$

In their terms the condition of deformations canonicity (14) may become

$$\tau^m(s) := \frac{d}{ds'} t^m(s, s'),\ \forall s,\ s' \in I,$$

where $\tau^m(s) = h(x(s))^m_{\mu}\tau^{\mu}(s)$ are components of vector in affine basis X_m which is tangent to curve in point $x(s)$, wherefrom follows condition for parameters of group Γ^{H}_{T}:

$$t^m(s, s') = (s' - s)\tau^m(s),\ \forall s,\ s' \in I, \tag{26}$$

equivalent to condition (14). Turning equation (25) with the use of property 2 H), this condition can have the following equivalent appearance:

$$\tilde{t}^\mu(s,s') = x(s')^\mu - x(s)^\mu = K^\mu(x(s),(s'-s)\tau(s)) \quad \forall s,\ s' \in I, \tag{27}$$

Thus, the curve for which condition (27) is fulfilled, according to the first part of Proposition 1 is geodesic passing through point $x(s)$ in direction of vector $\tau(s)$.

The opposite is true.

Proposition 3 (criterion of canonicity for geodesic equation).

If arbitrary geodesic in affine parameterization in chart O is written down as (27) with functions $K^\mu(x,t)$, defined by the deformation functions with property 2H), such deformations will be canonic.

This can be easily verified directly by fulfilling double differentiation of equation (27) with respect to s' (assuming $s = 0$ and redesignating s' as s) and demanding for coincidence of derivations with geodesic equation (15), which leads to equation (19) for deformation functions, providing for their canonicity.

Let's examine the product of elements of canonic deformed group Γ_T^H which correspond to shifts along geodesic and, therefore, have representation (26):

$$(s_1+s_2)\tau^m(s) = \varphi^m(x(s), s_1\tau(s), s_2\tau(s')), \tag{28}$$

where $s' = s + s_1$. This formula generalizes *canonic multiplication law* [3] for canonic finite parameter Lie groups for the case of infinite groups Γ_T^H , which accounts for our customary term "canonic" for groups Γ_T^H and respective deformations. Formula (28) defines homomorphism of the additive Abelian group $T^1 = \{s\}$ into group Γ_T^H .

Differentiating (28) with respect to s_2 in zero we derive

$$\tau^m(s) = \lambda(x(s), s_1\tau(s))^m_n \tau^n(s'), \tag{29}$$

which with consideration of formula (5) is equivalent to condition of canonicity of deformations (14).

Let's study the "left" analogue of condition (29). To this end let's differentiate (28) with respect to s_1 in zero. We derive as the result:

$$\tau^m(s) = \mu(x(s), s_2\tau(s))^m_n \tau^n(s) + s_2\tau^m(s), \tag{30}$$

Both equations (29) and (30) fix the curve along which canonic multiplication law (28) is fulfilled. Thus, differentiating in zero (29) with respect to s_1 , or (30) with respect to s_2, we derive the equation:

$$\dot{\tau}^m(s) + \gamma(x(s))^m_{kn}\tau^k(s)\tau^n(s) = 0, \tag{31}$$

where

$$\gamma^m_{kn} := \partial^2_{kn'}\varphi^m(x,t,t')|_{t=t'=0} = \partial_k\lambda(x,t)^m_n|_{t=0} = \partial_n\mu(x,t)^m_k|_{t=0}, \tag{32}$$

Insertion of expression (3) into the first equality of formula (32), with consideration of definition (8) results in $\gamma^m_{kn} = h^m_\mu(\Gamma^\mu_{k\nu}h^k_k h^\nu_n + h^k_k\partial_k h^\mu_n)$, accordingly, functions γ^m_{kn} are affine connection coefficients, while equation (31) is the equation of geodesic in affine vielbein.

Differentiating equation (30) with respect to s_2 and subsequently assuming $s = 0$ and substituting s_2 for s , with the use of equation (31) we derive the equation for auxiliary matrices $\mu(x,t)^m_n$:

$$[\partial_k\mu(x,t)^m_n - \gamma(x)^m_{kn}]\tau'^k\tau'^n = 0, \tag{33}$$

analogous to equation (18). Here according to (26) it is taken that $t(0,s) = s\tau$. Insertion of expression (6) of matrices $\mu(x,t)^m_n$ through deformation functions $H^m(x,\tilde{t})$ into equation (33) once again leads to equation (19) which provides for canonicity of deformations. Thus condition of deformations canonicity can be applied not only as equation (14) or equation (29) equivalent to it, but also as equation (30).

Consequent differentiation of equation (33) with respect to s in zero leads to the following condition for auxiliary matrices $\mu(x,t)^m_n$ of canonic deformed group Γ^H_T:

$$\partial^{(n)}_{\{k_1,\ldots,k_n}\mu(x,t)^m_{k_{n+1}\}}|_{t=0} = 0,$$

in particular

$$\partial^{(n)}_{\{kl}\mu(x,t)^m_{s\}}|_{t=0} = \rho^m_{\{kls\}} = 0$$

wherefrom directly follows expression (23) of coefficients ρ^m_{kls} through curvature tensor R^m_{kls}.

So, we have proved the following.

Proposition 4 (criterion of multiplication law canonicity).

Deformed group of diffeomorphisms Γ^H_T possesses canonic multiplication law (28) along geodesics if and only if it is canonic, for auxiliary matrices $\lambda(x,t)^m$ and $\mu(x,t)^m$ of group Γ^H_T equations (29) and (30) being fulfilled respectively.

4 Canonicity of Riemannian space deformations

In work [1] with requirement for vector length conservation during finite parallel transports for deformation functions in coordinate basis we derived equation (11) which uniquely defines them with respect to metric tensor $g(x)_{\mu\nu}$ including coefficients $\Gamma(x)^\sigma_{\mu\nu}$ in the second order of expansion with respect to $\tilde{t}$, which appear to be equal to Cristoffel symbols (12).

The canonicity of such deformations follows from the fact that equation (11) written down relatively to functions $\lambda(x, x' - x)^{\mu}_{\sigma}$:

$$\lambda(x, x' - x)^{\rho}_{\mu}\lambda(x, x' - x)^{\sigma}_{\nu}g(x)_{\rho\sigma} = g(x')_{\mu\nu}$$

provides for fulfillment of composition law (24) for them along geodesics, and, therefore, according to Proposition 2, such functions correspond to canonic groups Γ^{H}_{T}. This can be directly verified if equation (11) is differentiated with respect to $\tilde{t}$ and the result

$$\partial_{\tilde{\lambda}}g(x + \tilde{t})_{\mu\nu} = 2\partial^2_{\tilde{\lambda}\{\tilde{\mu}}H^{\rho}(x, \tilde{t})\partial_{\tilde{\nu}\}}H^{\sigma}(x, \tilde{t})g(x)_{\rho\sigma}$$

is inserted into expression (12), taken in point $x + \tilde{t}$. After cancellations we derive:

$$\Gamma(x + \tilde{t})^{\sigma}_{\mu\nu}g(x + \tilde{t})_{\sigma\lambda} = \partial^2_{\tilde{\mu}\ \tilde{\nu}}H^{\rho}(x, \tilde{t})\partial_{\tilde{\lambda}}H^{\sigma}(x, \tilde{t})g(x)_{\rho\sigma} \tag{34}$$

Using once again equation (11) for the expression of tensor $g(x + \tilde{t})_{\sigma\lambda}$ through derivatives from deformation functions and considering invertability of matrices $\partial_{\tilde{\lambda}}H^{\sigma}(x, \tilde{t})$ we find that equation (19) follows from equation (34), which provides for canonicity of transformations, functions of which satisfy the equation (11).

Let's point out that in Riemannian space geodesics in natural parameterization are extremals of energy functional

$$S(x, x') = \frac{1}{2}\int_0^s g(x(\alpha))_{\mu\nu}\tau^{\mu}(\alpha)\tau^{\mu}(\alpha)d\alpha. \tag{35}$$

The function of central field of extremals which start from each point x, if the function of action $S(x, x')$ which is defined by integral (35) with condition its extremality, satisfied the Hamilton-Jacobi equation [4]:

$$g(x')^{\mu\nu}\partial_{\mu'}S(x, x')\partial_{\nu'}S(x, x') = g(x)_{\mu\nu}\tau^{\mu}\tau^{\nu} \tag{36}$$

and allows to find vectors τ' tangent to geodesic in current point x' according to the formula:

$$\tau'^{\nu} = g(x')^{\mu\nu}\partial_{\mu'}S(x, x')$$

On the other hand, this vector may be found by means of parallel transport of initial vector τ to the point x' which yields:

$$\partial_{\mu'}S(x, x') = g(x')_{\mu\sigma}\partial_{\nu}H^{\sigma}(x', x - x')\tau^{\nu}.$$

Insertion of this relation into Hamilton-Jacobi equation (36) leads to fulfillment of equation (11) along geodesic, which can also be called Hamilton-Jacobi equation for deformation functions.

Thus, we have proved the following.

Proposition 5 (of canonicity of Riemannian space deformations).

Deformed groups of diffeomorphisms Γ_T^H which specify by their action in O the structure of Riemannian space, derived with the help of deformations, functions of which satisfy Hamilton-Jacobi equation (11), are canonic.

To crown it all, let's point out that finite parallel transports are naturally joined in the so called groups of parallel transports $DT = \Gamma_T^H \times) GL^g(n)$ $(DT = \Gamma_T^H \times) SO^g(n)$ in case of Riemannian space) [6], which acts in tangent bundle of space O and is a fundamental group (in terms of Klein's Erlangen Program) of space with affine connection (respectively to Riemannian space), while in case of canonic group Γ_T^H group DT has in the set of pure translations without additional rotations (which in case of curved space does not form subgroups of group DT) one-parameter subgroups, which translate two arbitrary points $x, x' \in O$ one into another. On the contrary, the existence of such subgroups in group of parallel transports DT fulfills canonicity of group Γ_T^H .

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